Can you Ever Truly Mix a Cuppa? – Spilling the Tea on Brouwer's Fixed Point Theorem – ILA by Ben Watkins

Introduction

Is it possible that there are always two places on earth with the same temperature and pressure? How does the game show Blockbusters have any implications on algebraic topology? Can a general equilibrium ever be reached in an economy? Perhaps most crucially of all, can you ever truly mix a cup of tea?

My ILA provides insight into Brouwer's fixed point theorem, a theorem found in the field of algebraic topology. It uncovers how a remarkable and seemingly counterintuitive result in what is often considered to be an abstract field of mathematics can have such broad and pertinent results in the real world. However, this isn't to say that this ILA doesn't uncover the result of this theorem for the sake of the beauty of it as much as uncovering it for the sake of its applications. Indeed, Luitzen Egbertus Jan Brouwer himself (the discoverer of this theorem as well as often being called 'the Father of Topology') was very much an upholder of this mentality: that maths has great importance for the sake of maths itself. Philosophically, Brouwer was a neo-intuitionist, which means that he thought of mathematics as purely a mental phenomenon, the result of constructive mental activity rather than uncovering any principles of an objective reality. He is often quoted in saying that "The construction itself is an art, its application to the world an evil parasite."

What is Brouwer's Fixed point theorem?:

The Brouwer's Fixed Point Theorem states that given a compact and convex set α that is the subset of some Euclidean space $\mathbb{R}^n$ for any value of $n \geq 1$, we can apply any continuous function f and there will always be some element, $c \in \alpha$ such that $f(c) = c$. Convex here means that for x, y in X, $cx + (1-c)y$ belongs to X for any $0 \leq c \leq 1$ i.e. given any two points that exist within the set, any point that exists along the line segment which connects these two points must also be a member of the set. Compact here means that all points within a set lie within some fixed distance of each other and that all the sets' limit points are self-contained within the set.

In simpler terms, what this theorem says that suppose that we have some object. Now, imagine this object as being made up by an infinite amount of points within the object. This object is bounded and doesn't have any holes in it. Now say that we can mould this object by stretching it, crumpling it, twisting it, and bending it but we aren't allowed to tear it or glue it. What Brouwer's fixed point theorem states is that at least one of those points within the object will stay in the same place that it started in, no matter how much we try.

For example, take two pieces of paper, leave one flat but take the other and crumple it as much as you desire and then place the crumpled piece of paper on top of the flat piece of paper. By Brouwer's fixed point theorem, we can say with certainty that at least one point of that crumpled piece of paper is directly above the point on the flat piece of paper that it would've been if we hadn't crumpled the piece of paper in the first place. The theorem doesn't say exactly where this point is but simply, that there is one.

The first part of my ILA shall cover how we can go about proving this to be true in both the 1-dimensional case and in the 2-dimensional case of a closed and bounded disk embedded in the $\mathbb{R}^2$ space. Then afterwards, I will delve into both major applications of this theorem for us in the real world as well as ending on some more trivial and novel facts that come about as a result of this theorem.

1 Dimensional Proof:

This proof comes in three stages:

1. Proving *Image of Interval by Continuous Function is Interval*
2. Proving *Intermediate Value Theorem*
3. Proving *Brouwer's Fixed Point Theorem in 1 dimension*

1. Proving *Image of Interval by Continuous Function is Interval*

Theorem: Let I be a real interval. *(A **real interval** is a range of numbers between two given numbers and includes all of the real numbers between those two numbers.)*

Let $f: I \to \mathbb{R}$ be a continuous Real Function. *(A **continuous function** is a function that does not have any abrupt changes in value such as in $y = 1/x$ for example)*

Then the image of f is a real interval *(an **image** of a function is the set of all output values it may produce)*

Proof: Let J be the image of f.

Suppose that we have two points $y_1, y_2 \in J$, and suppose that $\lambda \in \mathbb{R}$ where $y_1 \leq \lambda \leq y_2$.

Let $S = \{x \in I: f(x) \leq \lambda\}$ be the subset of the interval I where $f(x)$ is less than or equal to λ

Let $T = \{x \in I: f(x) \geq \lambda\}$ be the subset of the interval I where $f(x)$ is greater than or equal to λ

As $y_1 \in S$ and $y_2 \in T$, it suffices to say that $S, T \neq \emptyset$.

Also, $I = S \cup T$.

Let $s \in S$ be a point that is at zero distance from T. Such a point can exist as S and T are both closed and share a boundary.

Let $\langle t_n \rangle$ be a sequence in T such that:

$\lim_{n\to\infty} (t_n) = s$. *(the proof of the possibility of this is excluded but it is shown to be true by the 'Limit of Sequence to Zero Distance Point Theorem')*

Since f is continuous I: $\lim_{n\to\infty} (f(t_n)) = f(s)$.

But for all $n \in \mathbb{Z}^+ : f(t_n) \geq \lambda$. Hence we have that $f(s) \geq \lambda$.

But because $s \in S$, we have that $f(s) \leq \lambda$, as per our definition of S.

Therefore $f(s) = \lambda$ and so $\lambda \in J$.

(NB: the same can be done by considering a point t at zero distance from S.)

2) Proving the Intermediate Value Theorem

Theorem: Let $f: S \to \mathbb{R}$ be a real function on some subset S of $\mathbb{R}$.

Let $I \subseteq S$ be a real closed interval of S

Let $f: I \to \mathbb{R}$ be continuous on I.

Let $a, b \in I$

Let $k \in \mathbb{R}$ lie between $f(a)$ & $f(b)$.

That is, either: $f(a) < k < f(b)$ or $f(b) < k < f(a)$.

Then $\exists\, c \in (a..b)$ such that $f(c) = k$

(Effectively, this theorem states that on a continuous function, if you have two values on the function, then the function must contain every value on the range of these values in between these values.)

Proof

From *image on interval by continuous function is interval* we know that the image of $(a..b)$ under f is also a real interval.

Thus, if k lies between $f(a)$ & $f(b)$, it must be the case that $k \in img(a..b)$.

3) Proving *Brouwer's Fixed Point Theorem in 1 dimension*

Theorem:

Let $f[a..b] \to [a..b]$ be a real function on the closed and compact interval $[a..b]$ and let $a > b$ *(A **closed interval** is an interval that contains its endpoints so in this case, an interval which contains both a & b and everything real in between.)* (NB any subset of Euclidean space $\mathbb{R}^n$, that is closed is bounded according to *Heine-Borel* theorem, however this is evident in our case since it can be covered by an open interval $(a - 1, b + 1)$.)

Then there exists a point $c \in [a..b]$ such that $f(c) = c$.

Proof:

As the Codomain of f is $[a..b]$, it follows that $f \subset [a..b]$ *(The codomain is the bit after the arrow in the definition of a function.)*

Hence, $f(a) \leq a$ and $f(b) \geq b$. (statement 1)

Let $g: [a..b] \to \mathbb{R}$ such that $g(x) = f(x) - x$ (statement 2)

As both $f(x)$ and x are continuous functions, $g(x)$ is continuous on $[a..b]$ *(This isn't a necessary given, but it is intuitive and proven by the **combined sum rule for continuous functions.*** However, this isn't true for infinite sums of continuous functions as seen in Fourier Series, but we needn't concern ourselves.*)*

By combining statement 1 and statement 2, we can see that:

$g(a) \geq 0$ and $g(b) \leq 0$

Using the *intermediate value theorem, by letting k=0,*

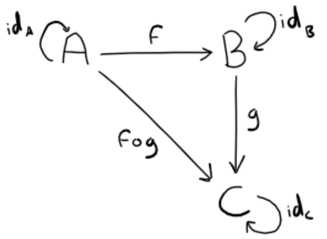
We know that there must exist a point $c \in [a..b]$ such that $g(c) = 0$.

Therefore, as $g(c) = f(c) - c$, there must exist a point where $f(c) = c$

An Introduction to Categories and Functors:

A **Category** in Mathematics refers to a group of objects that are linked by 'arrows' which symbolise morphisms between the objects. (A **Morphism** being another way to refer to a function, particularly between two different mathematical structures.)

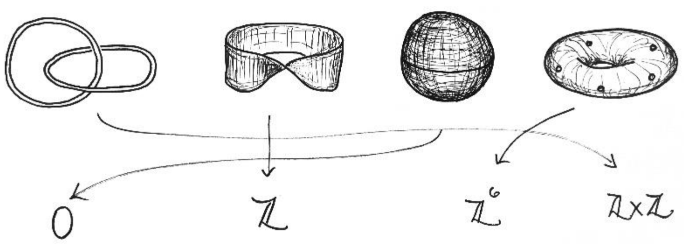
To be classed as a category, two basic criteria must be fulfilled: firstly, that arrows are associative and secondly that each object has an identity arrow, a morphism that returns itself. An example of a category is found below (Associativity is a property that given $\{G,*\}$ where G is a set and * is a function, and we have that $a, b, c \in G$, then $a * (b * c) = (a * b) * c$))



Suppose then, that we have two categories K & L, a functor $\boldsymbol{F}$ is a mapping (**mapping** being another term for function) from K to L such that:

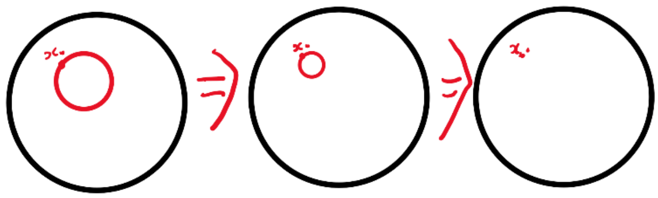
1) It associates every object x in K with an object $F(x)$ in L.
2) It associates each morphism in $f: A \rightarrow B$ in K a morphism $F(f): F(x) \rightarrow F(y)$ in L such that:
i) $F(Id_x) = Id_{F(x)}$ for every object x in K
ii)$F(g \circ f) = F(g) \circ F(f)$ for all morphisms $f: A \rightarrow B$ and $g: B \rightarrow C$.

One such functor $\pi_1: Top \rightarrow Group$ associates to every topological space T a group $\pi_1(T)$ called the Fundamental group of T. The fundamental group of T is defined as the group formed by the sets of equivalence classes of the set of all loops, (i.e. paths with initial and final points at a given basepoint.)

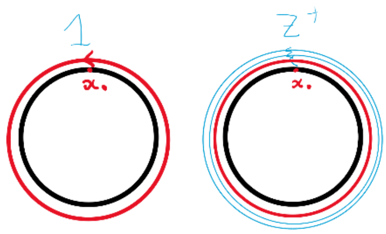


In topological parlance, π_1 is called a "Hole Detector" as the number of possible loops in any given topological shape is governed by the amount of holes in an object. This diagram here gives the values given by the functor for some topological shapes. Note that a sphere has no loops in it because any supposed loop on it would be topologically morphable to a point. (1)

(In topology only continuous deformations are allowed, such as stretching, twisting, crumpling and bending, but not tearing or gluing. Hence, points and circles on a sphere are topologically equivalent.)



Importantly for us, $\pi_1(S^1) = \mathbb{Z}$ (S^1 is the formal topological way of saying circle.) Whilst the formal proof of this is extensive[(2)], it's quite intuitive. Because given any point on a circle, you can get back to the same point by going around the circle clockwise an integer number of times or by going counter-clockwise an integer number of times.



Also, $\pi_1(D^2) = 0$ (D^2 is the formal topological way of saying a filled in circle.) Because topologically, there are no fundamental loops on a disk as each supposed loop can be shrunk down to a point, much like as in the case of the sphere without breaking the rules of topology and all loops are hence topologically equivalent to a point, which don't count as loops.

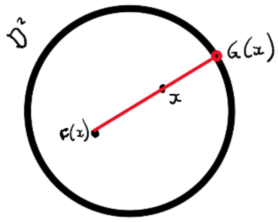
(1) Whitehead, George W. "On The Homotopy Groups of Spheres and Rotation Groups." Annals of Mathematics, vol. 43, no. 4, 1942, pp. 634–640. JSTOR, www.jstor.org/stable/1968956.
(2) http://pi.math.cornell.edu/~hatcher/AT/AT.pdf : Proof given by Allen Hatcher in his textbook 'Algebraic Topolgy,' an online version of which is cited here

2 Dimensional Proof:

Theorem: Let $F: D^2 \to D^2$ be a continuous mapping. Then there exists a point $x \in D^2$, such that $F(x) = x$

Proof by Contradiction:

Firstly, we assume that $F(x) \neq x$ for all $x \in D^2$. As such we can define a map $G: D^2 \to \partial D^2$ (where ∂D^2 is the boundary of D^2) that draws a straight-line through $F(x)$, passing through x, and meeting the boundary at $G(x)$. (The boundary of D^2 is the set of all points that lie on the edge of D^2, what might commonly be called S^1) Note that if $F(x) = x$ at any given point then this function couldn't exist because no trace would be possible for this point, however as we are assuming $F(x) \neq x$, we can have this function.



Evidently, $G(x) = x$ if $x \in \partial D^2$. As the straight-line would meet the boundary at x and we define the point at which the straight-line meets the boundary as $G(x)$. Here we note two properties of $G(x)$.

1) Each $G(x)$ only returns one value.
2) $G(x)$ is also a continuous function as a small change in x returns a small change in $F(x)$ which intuitively returns a small change in $G(x)$.

Let $i: \partial D^2 \to D^2$ be the inclusion map of the boundary. (i.e. i is the function which sends each element of x on ∂D^2 to x as treated as an element of D^2). That is to say that $i(x) = x$.

Note that we can compound these two functions, G and I to produce an identity formula:

$$G(i(x)) = G(x) = x$$

i.e. applying the compound formula to any point on ∂D^2 returns the same point on ∂D^2.

The next part of this proof is reliant upon taking the fundamental groups of each of these stages in this diagram:

$$\begin{tikzcd} \partial D^2 \arrow[r, "i"] \arrow[rr, bend right, "id_{S^1}"'] & D^2 \arrow[r, "G"] & \partial D^2 \end{tikzcd}$$

Applying this functor, we can see issue with the arrow diagram from before:

$$\begin{tikzcd} \pi_1(\partial D^2) \arrow[r, "\pi_1(i)"] \arrow[rr, bend right, "\pi_1(id_{S^1})"'] & \pi_1(D^2) \arrow[r, "\pi_1(G)"] & \pi_1(\partial D^2) \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} \mathbb{Z} \arrow[r, "i"] \arrow[rr, bend right, "id_{S^1}"'] & 0 \arrow[r, "G"] & \mathbb{Z} \end{tikzcd}$$

Notice that $\pi_1(i): Z \to 0$ Is a constant map. It assigns every integer to 0. Also, as we established, $G(x)$ only ever returns one value, in fact it returns 0 as $G(0) = 0$. Hence, if $G(x)$ were to be a valid function, it would imply that for all integers, $n = G(i(n)) = G(0) = 0$. Which would suggest that $n = 0$ for all $n \in \mathbb{Z}$. Hence no such function can exist. This means that there must be a point where x lies directly upon $f(x)$ such that no trace can be drawn. Therefore, for any function continuous mapping F, there exists a point $x \in D^2$, such that $F(x) = x$.

Important Applications

Despite Brouwer's neo-intuitionist approach to mathematics, his theorem has been used much in applied mathematics. Particularly, my ILA will discuss its impact on Economics and Game Theory.

Economics

Firstly, in economics, Brouwer's fixed point theorem has been used in the field of *General Equilibrium Theory*. Equilibrium in economics refers to the balance of supply and demand. Whilst *partial equilibrium theory* takes only into consideration a part of the market, typically single markets, *general equilibrium* considers the totality of the economy where there are several interacting markets and seeks to prove that eventually, supply and demand resolve to an equilibrium. Historically, this theory dates to the 1870s work of French mathematician León Walras '*Elements of Pure Economics'* and his theory holds crucial importance to modern-day economies.

General Equilibrium, if achieved would be an economic utopia. It has been repeatedly shown that economies become increasingly stable the closer that they come to a general equilibrium. General equilibrium leads to the stability of prices. As prices become more stable, this means that both businesses and consumers can make long-term financial plans. This is a good position that any economy would like to be in.

The problem that we are concerned with, however, is whether General Equilibrium is even possible, whether its existence is attainable. Walras did provide proofs. However, his proofs were insufficient for the non-linear systems of equations that regularly crop up in supply and demand theory. This was where Lionel McKenzie stepped in. In his 1959 work *"On the Existence of General Equilibrium for a Competitive Economy",* he utilised Brouwer's fixed point theorem in order to prove the existence of these General Equilibriums. [(3)] His proof, in summary, uses the idea of a "Utility Function" U (which models the individuals' preferences) acting upon a bundle of goods $\{x_1, x_2 \dots x_n\}$ then by modelling preferences as monotopic (more is better), he demonstrates how consumer preferences are convex, and from that he shows how an equilibrium (or fixed point) has to exist as a result of Brouwer's fixed point theorem.

(3) Mitra, Tapan, and Kazuo Nishimura, editors. "On the Existence of General Equilibrium for a Competitive Market." *Equilibrium, Trade, and Growth: Selected Papers of Lionel W. McKenzie*, by Lionel W. McKenzie, The MIT Press, Cambridge, Massachusetts, London, England, 2009, pp. 101–122. *JSTOR*, www.jstor.org/stable/j.ctt6wpxws.8. Accessed 11 May 2020

Game Theory

Brouwer's Fixed point theorem also has great use in game theory, especially in the proof of what is referred to as Nash's theorem, named after the father of Game Theory: John Nash. Game theory is concerned with the playing of, and strategies that lead to getting good outcomes from games. A game is defined as a situation whereby multiple 'players' make decisions towards a result which is predicated by the set of circumstances at play. Strategies are hence defined as plans of action that a player will take given the set of circumstances. An optimal strategy doesn't necessarily mean that a player is guaranteed the best outcome every single time but rather that a player would not be able to expect a better outcome of the game by changing their strategy. A Nash's equilibrium is when *all players have a strategy such that even when each player gets to see the other players' strategies, they don't change their strategy.*

For example, suppose that two 'players' are playing a game where they get a decision between two options: option A and option B. If a player chooses option A, they win £100, if they choose option B, they lose £50. Here the obvious strategy is choosing option A every time, and even if you reveal your strategy to your opponent, they would carry on choosing option A. A Nash equilibrium exists quite obviously for this game. However, game theory is often interested in more interesting games, for example, could an equilibrium be reached if both players lose £100 when they both pick option A?

Remarkably, what John Nash proved was that every n-player game with a finite number of pure strategies has a Nash equilibrium. He accomplished his proof in a similar way that Lionel McKenzie was able to construct a proof for General Equilibriums. To briefly sum it up, by supposing each game to have i players $i \in \{1, \dots, n\}$, each of whom has A_i strategies then each player has a utility function $u_i: A \rightarrow \mathbb{R}$ which considers the sum of each possible set of strategy 's utility multiplied the probability that each combination will be played. The combination of these utility functions must necessarily produce some fixed point by Brouwer's fixed point theorem. Hence, every n-player game with a finite number of pure strategies has a Nash equilibrium.

In our example game, an equilibrium can be reached by both players going for an alternating approach, choosing A and then B whilst the other player chooses B and then A. This results in each player gaining £50 every second go. This is a Nash's equilibrium because even if the two players were to reveal their strategy to their opponent, they would still follow the same strategy. After all, it provides the optimal outcome for both of them.

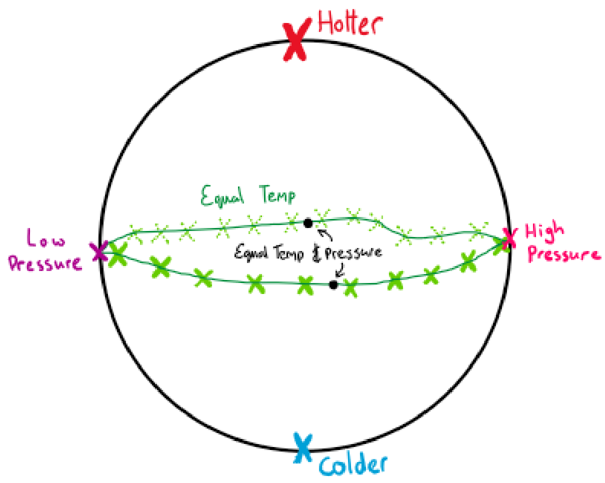
Some Interesting Conclusions:

Whilst not being particularly crucial to the modern world, there are some perhaps trivial results of Brouwer's fixed point theorem which are particularly interesting. Here are five such cases.

1) A Cup of Tea
   If you mix a cup of tea with a spoon, you are in effect, applying a continuous function to a convex object in 3D Euclidean Space. Hence, no matter how much you try, some part of that tea will always end up in the same place that it started in. (This is of course if we remove the discrepancies caused by tea ultimately being made up of particles rather than an infinite number of points, but this fact is still approximately true.)

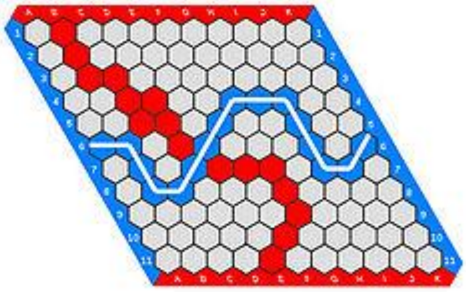
2) Two DJs
   Suppose you have two DJs. The first one plays a song exactly as it was brought out, with no altering. The second one warps the song, speeding it up in places perhaps to alter the pitch and slowing it down at other times. As long as the first DJ starts before and finishes after the second DJ, it is necessarily true that at least one point in the song, the two DJs will be playing exactly the same part of the song. We can imagine the second DJ as having done a continuous function to a one-dimensional object by warping the song. Hence, Brouwer's fixed point theorem applies.

3) Meteorology
   At any given time, there are at least two points on the Earth where the temperature and atmospheric pressure at those points are entirely identical. A good way of imagining this is to picture two points on the Earth's surface with different temperatures. As one moves from the hotter temperature to the colder temperature, as temperature is gradient, at one point, the temperature must be exactly equal to the average of the two temperatures. Now, one can make this journey in several straight lines around the planet and connect the points where the temperature is the same. Now, we can pick two points on this line that has been created, suppose that they have different pressures, starting at the higher pressure, we can move to the lower pressure and at one point the pressure, as pressure is gradient, must be exactly the average of these two pressures. You can also walk around this line the other way to find another point. Hence, these two points have the same temperature and pressure.



4) <u>Cartography</u>
If you take a map of any place in the world, say Wales, and make a copy of it. Then you apply a continuous function to the map, enlarge it, rotate it or, even crumple it if you want to. Then we know for a fact that if we take that map and put it on top of the other map, one location on that map will rest on top of the same location on the other. What's more, if we take that map to Wales itself, some point of that map will lie exactly on top of where that point actually is in Wales.

5) <u>The Game of Hex</u>
If you've ever watched an episode of Blockbusters, you'll be familiar with this game. It was invented by a Danish engineer in 1942 and thoroughly investigated in Princeton by John Nash in 1948. The game involves two players. One plays vertically and the other plays horizontally. The vertical player needs to make a chain of connected hexagons that connects the top to the bottom. The horizontal player needs to do the same from one side to the other. The two players alternate turns. In the example below, the horizontal player is using blue tokens and has won by making a complete chain.



John Nash demonstrated that it is impossible for a game of hex to result in a draw. This can be thought of simply because the only way to fully block your opponent is to complete your own chain, which would result in your own victory. This is known as in combinatorics as the *Hex Theorem*.

In 2008, David Gale, American mathematics and economics professor at the University of California, Berkeley, used this 'Hex theorem' to produce a combinatorial proof for Brouwer's fixed point theorem.[(4)] As Francis Su, a Harvey Mudd College mathematician puts it "Hex implies Brouwer's!"

(4) *2008 The Game of Hex and the Brouwer Fixed-Point Theorem by David Gale American Mathematical Monthly*, vol. 86, 1979, pp. 818-827

<u>A Brief Final Thought</u>

I think that ultimately, I partially disagree with Brouwer's pessimistic neo-intuitionist way of looking at Mathematics. I think that what my ILA shows is how even the most seemingly abstract fields of mathematics (as algebraic topology is often considered to be) seem to be applicable to a spectrum of situations. For this theorem, in particular, it is able to relate not only to the physical world but also to economics and game theory. And further to this, despite its utility, the proofs that underpin the mathematics can still be quite beautiful.

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