

Will Knot Theory Pave Our Future?

ILA

Thomas Dowson

May 2024

Contents

1	Introduction	1
2	Origins	2
2.1	Gauss Code	2
3	Key Definitions	3
3.1	Prime Knots	4
3.2	Variants	4
4	Reidemeister moves	5
5	Tricolourability	6
6	Jones Polynomial	7
7	Modern uses	8
7.1	Biological DNA	8
7.2	Quantum computing	8
8	Summary	9

1 Introduction

A knot is a simple yet complex object, which can be found almost anywhere in the world. If you have ever been climbing I'm sure you can appreciate the power a knot has. Can a knot be more than just a rope used as a safety measure for people who can't climb?

If we look at a knot, surely there must be some way of describing it. Looking at the dictionary definition it states 'a join made by tying together the ends of a piece or pieces of string, rope, cloth, etc'. In comes Knot theory, a constantly developing branch of mathematics and physics. Where advances are being constantly made in the quantum branch with new knot variants and invariants being discovered, and quantum fields and gravity being developed consequently. Although quantum might seem unfamiliar to many, knot theory finds applications across various fields of science. Both computer science and mathematics feature specialized branches dedicated to the study of knots.

2 Origins

Knot theory was never actually intended for certain branches of quantum fields or even mathematics, rather chemistry. Where in the latter half of the 19th century, Lord Kelvin suggested that atoms would consist of knotted rings of an ether like substance where different elements had different knots. This idea came to mind after watching smoke rings floating in the air, as was the usual from the pipes which were commonplace at the time. This theory would hopefully enlighten scientists on how different elements would absorb and emit light at different wavelengths. The early proof, in 1860, came from the spectrum lines visible from the element. Spectral lines are produced as each element has its own energy, and will produce photons of distinct energy levels which appear as colours at distinct levels. Knot theory would hopefully explain why each element will have a unique 'fingerprint' of spectral lines. Sadly while these theoretical possibilities sounded logical, when tested these did not end up being the case. Even though he was not first person to take a leap forward into the mathematics of knots, Kelvin was the first person to aim to tabulate these findings. [Sil06]

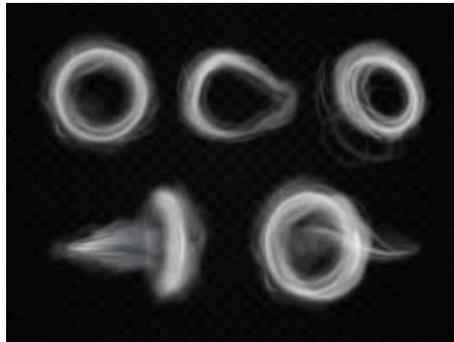


Figure 1: Smoke rings simulating knotted ether

2.1 Gauss Code

The first person to make progress into the mathematics behind knots was Carl Friedrich Gauss in the early 1800's. His early theories involved the following idea.

1. Arbitrarily choosing a point on a line.
2. Imagine you are walking the knot, labelling the first crossing (the point on a knot where two strands overlap) as 1, and any future crossings you reach as 2, 3, and so on, until you get back to where you start.
3. To record the Gauss code of the knot, you simply walk the knot again, this time noting down each crossing you reach, and if your strand goes over another, note down as the positive value of the crossing, and if it goes under another, it will be recorded as negative.

An example of the Gauss code can be seen below in Figure 2
Extended Gauss Code

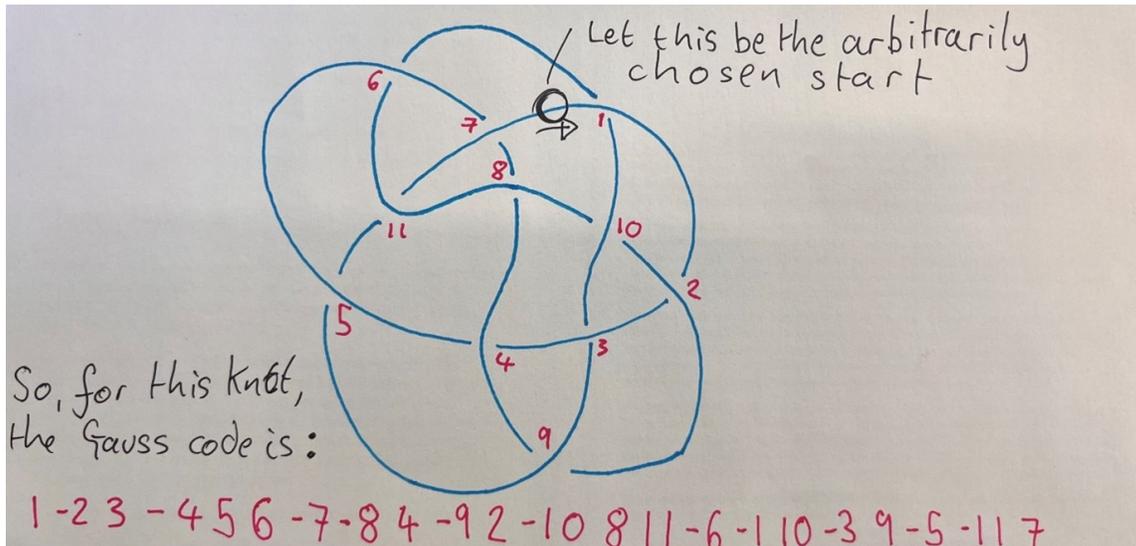


Figure 2: Example of a knot with its Gauss code

However, Gauss never realised the potential this branch of mathematics he had discovered held and left his research with the extended Gauss code, which was able to further describe the construction of the knot by giving the positive and negative values depending on the nature of the crossing. The nature of the crossing being if when it passes underneath or over, whether or 'knot' it is a right handed crossing or a left handed one. This is shown in Figure 3.

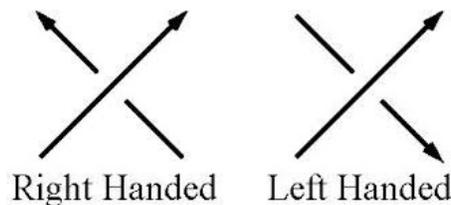


Figure 3: Depicting the two forms of crossings

This can lead to a more complex code for each given knot but with each code being more complex, the benefit gained is when analysing the code, it is possible to recreate the knot to a more exact form than with the simple Gauss code. [Bre06]

3 Key Definitions

As Knot theory developed, certain key concepts were defined, namely being prime knots and variants which are still key to modern day developments in knot theory.

3.1 Prime Knots

A prime knot holds a status similar to prime numbers.

This is a knot which can not be decomposed into any simpler knots the the knot sum process. The knot sum process is where you take to knots and by cutting each knot open at any point and then joining the ends to any other knot where the same process has taken place. This creates a new and more complex knot which combines the properties of the two individual knots.

All of the Prime knots can be seen below up to the seventh crossing but if you expand this list up to the sixteenth crossing you get 1701936 different prime knots. [Lic81]

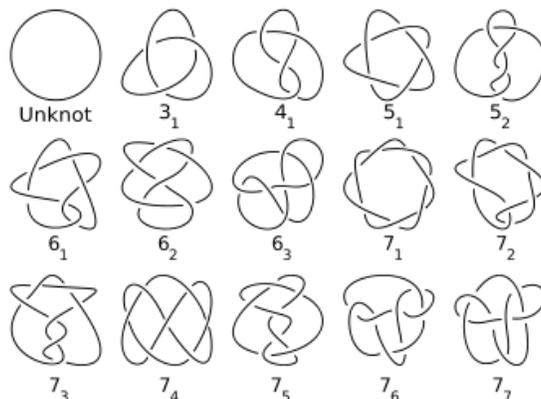


Figure 4: Showing all Prime Knots up to the 7th crossing

3.2 Variants

Variants are distinct shapes that one can create by distorting a particular knot in it's form. For example, take any general knot, for instance the trefoil. By altering the way it twists or loops, or even adding in new twists and loops, you might be able to get the same basic knot in a different form. Many these variants can look drastically different, though they are all are based on topologically the same starting knot.

One other way this is similar is in fractions where $\frac{2}{3}$ is the same as $\frac{4}{6}$. These are variants of each other but we know that these two fractions are the same. This is the same principal in Knot theory where the unknot is the same as the Goeritz unknot while looking completely different. [Wu92]

For another example, each time you take any simple knot and add just one extra loop or twist, you generate a new variant of that original knot. Variants, in this way, give tools by which mathematicians understand all possible configurations of the shape a single knot can assume. This teaches mathematicians about the fundamental properties of knots and how they compare to each other. [PW10]

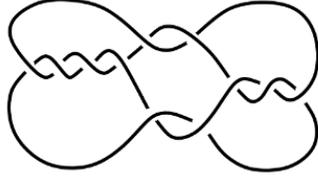


Figure 5: The Goeritz unknot

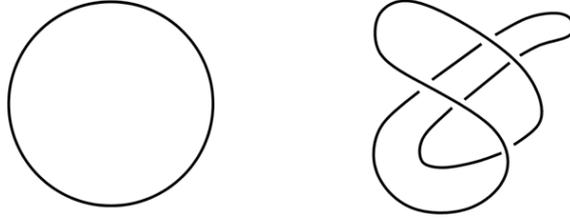


Figure 6: An example of the unknot and another of the unknot variants

4 Reidemeister moves

In 1927, the basis of knot theory as its known today was discovered. Kurt Reidemeister wanted to prove that knots do exist that are distinct from the unknot. The unknot can be thought of as a single loop with no crossovers or alternatively a slack elastic band. He went on to develop a series of 3 moves which could be completed without changing the underlying topology of the knot. These moves are local transformations which are as follows:

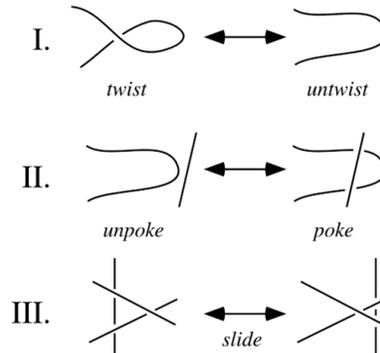


Figure 7: Depicting the 3 Reidemeister moves

With the development of these moves, he was able to identify whether a knot is distinct from another knot. However at this stage, this required many hours of trial and error completing the calculations. Reidemeister used these moves he had discovered which do not change the underlying topology of the knot to rigorously prove that there were knots that were different to the unknot.

Which may sound trivial but it opened up the world of knot theory for future mathematicians to develop. [Tra83]

5 Tricolourability

Theories began developing from knot diagrams to other ways to differentiate knots. A knot diagram being the simplified representation of a knot drawn on a single plane and each knot strand is represented as continuous curves. The key rule of tricolourability is that no two sections that share a common edge should be assigned the same colour. A section of a knot being from where if you follow the knot from where it goes under the last crossing to the next point it goes under the next crossing, as shown in Figure 5 where each section of the knot is coloured using red, green, and blue. [Alm12]

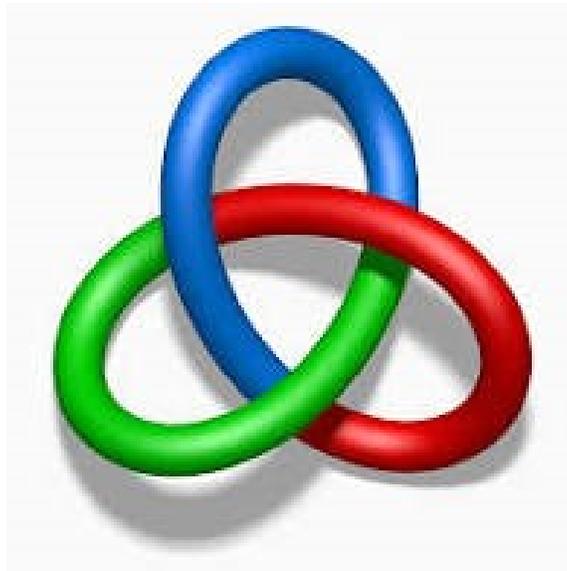


Figure 8: Example of a tricolourable trefoil

There are knots that are not tricolourable, such as the figure-eight knot. The reason tricolourability is so useful, is that this process can be used to serve as a variant for distinguishing distinct knots. This is because the Reidemeister moves, which as stated earlier, do not change the topology of the knot. And as each of the 3 Reidemeister moves are tricolourable, if the prime knot is tricolourable, all other variants will be tricolourable as well. [BH15]

This makes tricolourability a very useful technique for comparing knots. This simple process aids in categorizing and developing the subject of knot theory. While also allowing people to easily see a visual difference without having to do complicated mathematics. [Azr12]

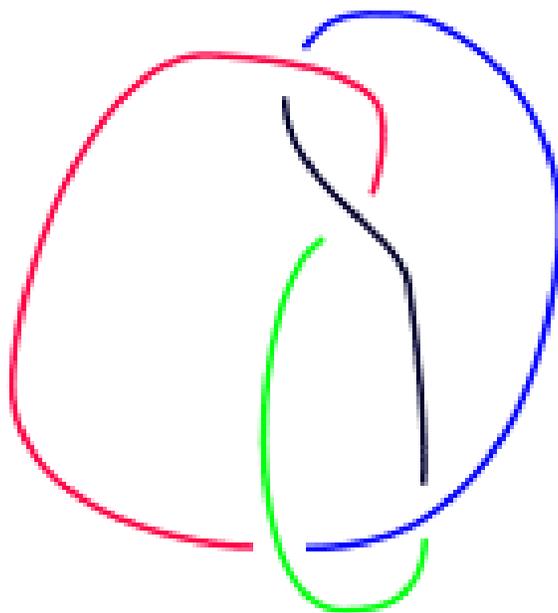


Figure 9: Example of a non-tricolourable figure of eight knot

6 Jones Polynomial

The Jones polynomial is a relatively new discovery in the realm of knot theory which adds a unique polynomial number to each knot or link, from which they are sorted into different categories. The polynomial was invented by Vaughan Jones in the 1980s and has become one of the most powerful tools in modern knot theory. [Jon05]

To understand the Jones Polynomial better, It's important to explain what a knot diagram is again. A diagram is nothing but a series of crossings: at each crossing, one of the 'threads' can go over or under another 'thread'. With this at our disposal we can then compute the Jones polynomial for the given knot.

For one of the simplest knots, the trefoil, where the given polynomial is $V(t)$. Which is $V(t) = t^2 - t + 1$, where t is simply a representation of the knot diagram. However this is not the important part of the equation, what matters are the powers which show the crossings. If we compare this to the unknot which has a Jones polynomial of $V(t) = 1$, comparing $V(t) = t^2 - t + 1$ to $V(t) = 1$, we can clearly see that they are different. [Jon14]

This is the power of the Jones Polynomial, it is able to tell the difference between things that, from an initial glance, look the exact same. But have different Jones polynomials. It is likely that two knots with the same Jones polynomial are the same topological knot, but further checks are needed to be certain.

But to summarise, imagine the Jones polynomial as a signature for knots which is each individual polynomial. This allows mathematicians to tell the knots apart and to examine them more easily. [Big02]

7 Modern uses

7.1 Biological DNA

Now we can somewhat understand the way a knot is created and categorised, how could what seems to be a purely theoretical concept be able to save a life?

If we take a look at biology, specifically at the structure of DNA, the classic double helix. To create human life, the amount of data required to be stored on DNA is massive. Requiring these strands of DNA to be up to 2m long. To sustain human life, this DNA needs to be constantly produced and as it is produced these long strands inevitably get knotted up. Take any cables, when left in a box, will inevitably get knotted up. Developments have been made recently where we are now able to autonomously analyse and unknot a knot using a robot. this uses the principles of knot theory to predict what will happen when a certain action is performed. [VSK⁺22]

Now using these principles modern scientists are able to analyse strands of DNA and where required, unknot segments and in other cases create knots. By recreating natural knots in the DNA which may have been broken certain functions will be restored. As the shape is important for binding sites in the cell, damaged DNA may be life threatening and this new research can help save lives by fixing this DNA or creating new strands. [LJ15]

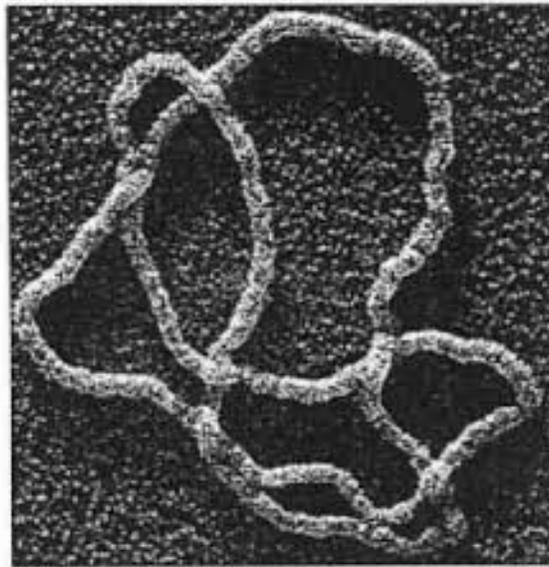


Figure 10: Example of a knotted loop of DNA

7.2 Quantum computing

Quantum computing is what the world is moving towards as we require computers to handle larger and more complicated questions. Quantum computing is based on qubits that can coexist in many steps allowing these computers to deal with many complex problems at the same time. This technology is currently used for breaking encryption methods. There are two big problems, namely

the fact that it is very expensive and there is a lot of hardware required but also that qubits are easily disturbed in the environment. [Col06]

The Topological computer aims to solve these problems using braided anyons. This significantly reduces the likelihood of errors forming and the calculations can be done faster using polynomial time not euclidean time.

However precise manipulation is required when dealing with anyons, as only recently was there experiments proving they exist. However further developments are needed to understand the structure and how to layer them in computers. [MMM⁺18]

By using knot theory in the building of quantum computers we get the fact that: knot theory is being used to build computers which are used to find new variants for knot theory, developing the subject while making technological advances.

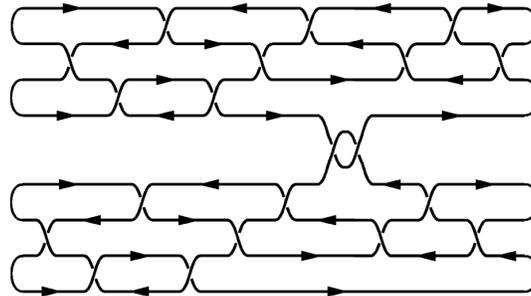


Figure 11: An example of a knot which can be braided by anyons to be used in a topological quantum computer

8 Summary

Knots are more than just objects in a physical world, but instead concern a complicated study in both mathematics and science. From initial hypotheses by people like Kelvin and Gauss has developed into this sophisticated discipline which is knot theory. The tools we need are given to us by the work of the Jones polynomial and the Reidemeister moves allowing us to classify and manipulate knots.

From this abstract branch of mathematics modern uses span into many areas for example biology and chemistry. But majorly quantum computing where knot theory is the key in developing more advanced ideas. As more research develops, further links with knot theory are being made and why the study of knot theory is so important to our future.

References

- [Alm12] Manuela Almeida. Knot theory. *Estados Unidos*, 2012.
- [Azr12] M Azram. Knots and colorability. *Australian Journal of Basic and Applied Sciences*, 6(2):76–79, 2012.

- [BH15] Danielle Brushaber and McKenzie Hennen. Knot tricolorability. 2015.
- [Big02] Stephen Bigelow. A homological definition of the jones polynomial. *Geometry & Topology Monographs*, 4:2941, 2002.
- [Bir93] Joan S Birman. New points of view in knot theory. *Bulletin of the American Mathematical Society*, 28(2):253–287, 1993.
- [Bre06] Felix Breuer. *Gauss codes and thrackles*. PhD thesis, Citeseer, 2006.
- [Col06] Graham P Collins. Computing with quantum knots. *Scientific American*, 294(4):56–63, 2006.
- [DF87] MJ Dunwoody and RA Fenn. On the finiteness of higher knot sums. *Topology*, 26(3):337–343, 1987.
- [Jon05] Vaughan F.R. Jones. The jones polynomial. *University of California*, 2005.
- [Jon14] Vaughan Jones. The jones polynomial for dummies. *University of California Berkley*, 2014.
- [Lic81] WB Lickorish. Prime knots and tangles. *Transactions of the American Mathematical Society*, 267(1):321–332, 1981.
- [LJ15] Nicole CH Lim and Sophie E Jackson. Molecular knots in biology and chemistry. *Journal of Physics: Condensed Matter*, 27(35):354101, 2015.
- [Man18] Vassily Olegovich Manturov. *Knot theory*. CRC press, 2018.
- [MK96] Kunio Murasugi and Bohdan Kurpita. *Knot theory and its applications*. Springer, 1996.
- [MMM⁺18] D Melnikov, A Mironov, S Mironov, A Morozov, and An Morozov. Towards topological quantum computer. *Nuclear Physics B*, 926:491–508, 2018.
- [PW10] Peter Pagin and Dag Westerståhl. Compositionality i: Definitions and variants. *Philosophy Compass*, 5(3):250–264, 2010.
- [Sil06] Daniel S Silver. Knot theory’s odd origins: The modern study of knots grew out an attempt by three 19th-century scottish physicists to apply knot theory to fundamental questions about the universe. *American Scientist*, 94(2):158–165, 2006.
- [Tra83] Bruce Trace. On the Reidemeister moves of a classical knot. *Proceedings of the American Mathematical Society*, 89(4):722–724, 1983.
- [VSK⁺22] Vainavi Viswanath, Kaushik Shivakumar, Justin Kerr, Brijen Thananjeyan, Ellen Novoseller, Jeffrey Ichnowski, Alejandro Escontrela, Michael Laskey, Joseph E Gonzalez, and Ken Goldberg. Autonomously untangling long cables. *arXiv preprint arXiv:2207.07813*, 2022.
- [Wu92] FY Wu. Knot theory and statistical mechanics. *Reviews of modern physics*, 64(4):1099, 1992.